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DECOMPOSITION OF STRAIN MEASURES AND THEIR RATES IN FINITE DEFORMATION ELASTOPLASTICITY

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Abstract—For large deformation elastoplasticity, the question of the decomposition of the total strain and the strain rate into the elastic and plastic constituents is critically examined, a new decomposition is introduced, and some existing misinterpretations and errors are corrected.

1. INTRODUCTION

In the classical small deformation elastoplasticity theory the infinitesimal strain tensor $\epsilon_{ij} = (1/2)(\partial u_i/\partial x_j + \partial u_j/\partial x_i)$ is unambiguously divided into the "elastic" and "plastic" parts additively as $e_{ij} = \epsilon^e_{ij} + \epsilon^p_{ij}$, where x_i, $i = 1, 2, 3$, denote the rectangular Cartesian coordinates, and u_i are the displacement components. Here, no distinction needs to be made between the Eulerian and the Lagrangian variables. Hence, partial time differentiation yields $\dot{\epsilon}_{ij} = \dot{\epsilon}^e_{ij} + \dot{\epsilon}^p_{ij}$, which is the decomposition of the strain rate tensor. In these decompositions the plastic part of the strain, or that of the strain rate, is defined on *physical grounds by elastic unloading which should involve no additional plastic flow.*

In finite elastoplastic deformation problems, the decomposition of the strain measures or that of their rates is not a clearcut nor an unambiguous task. This has led to some controversy and some misinterpretations in recent years. Part of the difficulty rests on the fact that for finite deformation, one may use different measures of strains and strain rates. The choice is more a matter of taste and convenience than anything else. Moreover, a measure of strain that submits to a convenient decomposition may result in an inconvenient decomposition for its rate.

It is the purpose of this paper to examine systematically various decompositions, to develop the relationship between them, and to introduce some new decompositions which may be more convenient for various applications. In this manner, the subject is critically reviewed, and some common misinterpretations and errors are corrected.

2. STATEMENT OF PROBLEM

Let a body $\mathscr{B}$ be deformed from its initial undeformed (stress free, virgin state) configuration $\mathscr{C}_0$ to a current configuration $\mathscr{C}$, and denote the corresponding one-to-one mapping by†

$$\mathbf{x} = \mathbf{x}(\mathbf{X}, t) \quad \text{or} \quad x_a = x_a(\mathbf{X}, t), \quad a = 1, 2, 3, \tag{2.1}$$

which defines at time t particle positions x_a in $\mathscr{C}$ in terms of their initial coordinates X_A, $A = 1, 2, 3$, both taken with respect to a fixed rectangular Cartesian coordinate system with unit base vectors $\mathbf{e}_a$. Denote the matrix† of the deformation gradient $\partial x_a/\partial X_A$, by F, and assume that $0 < \det F < \infty$. For measures of deformation, one may use either Green's deformation tensor or the Lagrangian strain tensor, which, respectively, have the following matrix representation:

$$C = F^T F, \quad E = \frac{1}{2}(C - I), \tag{2.2}$$

where I is the identity matrix.

If $\mathbf{v} = v_a \mathbf{e}_a$ is the velocity field expressed in terms of the current particle positions, i.e.

†Vectors are denoted by bold-face letters and matrices are denoted by upper case italic letters.

$\mathbf{v} = \mathbf{v}(\mathbf{x}, t)$, the deformation rate tensor may be represented in matrix form by

$$D = \frac{1}{2}[\partial v_a/\partial x_b + \partial v_b/\partial x_a] \equiv [D_{ab}]. \tag{2.3}$$

Within the framework of the continuum approximation, a class of materials is often considered, for which any general (macro) deformation can be regarded to be "locally" homogeneous (in the macroscopic sense). For this class of materials, the deformation and rotation of a given material neighborhood is assumed to be completely defined by the deformation gradient F evaluated at a typical point *inside* of this neighborhood. In a *microscopic* scale, however, even for this class of materials, there may exist intense local inhomogeneities in the material distribution and the deformations. [The microscale need not be, and for our purposes, in general, it is not, of atomic or molecular dimensions.] It is such intense local inhomogeneities in the microscale that give rise to macroscopic inelastic material behavior.

A class of materials of this kind is called "elastroplastic" in the sense that upon "unloading"† from configuration $\mathscr{C}$, only a certain part of the strains would be recovered. In the literature one often finds discussions of a class of elastoplastic materials which, if initially homogeneous and if subjected to a homogeneous deformation, then upon unloading the "elastic" part of the deformation recovers *without any additional plastic flow*, and there remains the "plastic" part of the deformation. In the present work, we shall consider only this restricted class of elastroplastic materials. We note, however, that even if the body is initially globally (i.e. macroscopically) homogeneous, and even if it undergoes a homogeneous deformation, still it may possess such a substructure that its unloading involves some additional plastic flow.‡ Some of our results do not apply to this class of materials.

When a material body is not globally homogeneous, or when it does not undergo a homogeneous deformation (or both), we may proceed as follows. Consider a particle $\mathbf{X}$, and let $N(\mathbf{X})$ be such a small material neighborhood of $\mathbf{X}$ that it, together with its deformation, can be regarded as (macroscopically) homogeneous. $N(\mathbf{X})$ in $\mathscr{C}_0$ is mapped to $n(\mathbf{x})$ in $\mathscr{C}$. Isolate $n(\mathbf{x})$ and apply all surface tractions which represent the effect of the remaining part of $\mathscr{B}$ in configuration $\mathscr{C}$ upon $n(\mathbf{x})$; apply also the body forces. Release these tractions and forces, and observe that this unloading involves no additional plastic flow (for the restricted class of materials considered here). If this process is performed for all material neighborhoods which comprise the body, one obtains an intermediate configuration $\mathscr{C}_p$ which is "incompatible" in the sense that various material neighborhoods do not "fit" into an unstressed monolithic body. Nevertheless, from the standpoint of describing the material properties, one may consider each small material neighborhood deparately. For this reason, and because the homogeneous deformations of homogeneous bodies§ lead to compatible intermediate states (i.e. the state obtained after unloading), we will consider in Sections 3 and 4 this special case, and then extend our results to more general (macroscopically nonhomogeneous) deformations in Section 5.

Let $\mathrm{d}\mathbf{X}$ be a material line element in $N(\mathbf{X})$, which has the length $\mathrm{d}S$ in the initial configuration $\mathscr{C}_0$, the length $\mathrm{d}s$ in the current configuration $\mathscr{C}$, and the length $\mathrm{d}s_p$ in the intermediate configuration $\mathscr{C}_p$. According to Lee[1], Mandel[2] and others, the total stretch, the "plastic" stretch and the "elastic" stretch for this element are given, respectively, by

$$\Lambda = \frac{\mathrm{d}s}{\mathrm{d}S}, \quad \Lambda_p = \frac{\mathrm{d}s_p}{\mathrm{d}S}, \quad \Lambda_e = \frac{\mathrm{d}s}{\mathrm{d}s_p} \tag{2.4}$$

so that one has the following decomposition:

$$\Lambda = \Lambda_e \Lambda_p. \tag{2.5}$$

The decomposition in (2.5), although logically correct for the restricted class of materials

†That is when all applied forces and constraints are released.
‡This fact was pointed out to the author by Dr. R. Hill in a private communication.
§This is certainly the case for the restricted class of elastoplastic materials considered here.

considered, is formal and of limited usefulness, because it is rather awkward to consider in the course of finite elastoplastic deformation of a solid, a *total* unloading at each instant. Most calculations of problems in finite elastoplasticity are done *incrementally* (see for example, Refs. [3–6]).

For an incremental calculation one requires a "*rate*" formulation. The decomposition (2.5), however, leads to an awkward rate formulation. To see this, we take the material time derivative of both sides of (2.5), and then multiply the results by Λ^{-1}, to obtain,

$$\frac{\dot{\Lambda}}{\Lambda}=\frac{\dot{\Lambda}_e}{\Lambda_e}+\frac{\dot{\Lambda}_p}{\Lambda_p}. \tag{2.6}$$

Lee[1] identifies the first term in the right side of (2.6) as the "elastic" part of the stretch rate and the second term in the right side as the "plastic" part of the stretch rate. *While the last term in* (2.6) *is, in fact, the plastic stretch rate measured per unit length in the intermediate configuration* $\mathscr{C}_p$, *the first term in the right-hand side is not a proper measure of the elastic stretch rate, as it involves also the plastic rate of deformation.* To see this, we note that while $\mathrm{d}S$ is a material length, $\mathrm{d}s$ and $\mathrm{d}s_p$ are not (i.e. $(\mathrm{d}S)^{\cdot}=0$, but $(\mathrm{d}s)^{\cdot}\neq 0$ and $(\mathrm{d}s_p)^{\cdot}\neq 0$, the superposed dot denoting the material time derivative). Hence, from (2.4) and (2.6) one obtains,

$$\frac{\dot{\Lambda}}{\Lambda}=\frac{(\mathrm{d}s)^{\cdot}}{\mathrm{d}s},$$

$$\frac{\dot{\Lambda}_p}{\Lambda_p}=\frac{(\mathrm{d}s_p)^{\cdot}}{\mathrm{d}s_p},$$

$$\frac{\dot{\Lambda}_e}{\Lambda_e}=\frac{(\mathrm{d}s)^{\cdot}}{\mathrm{d}s}-\frac{(\mathrm{d}s_p)^{\cdot}}{\mathrm{d}s_p}=\frac{\dot{\Lambda}}{\Lambda}-\frac{\dot{\Lambda}_p}{\Lambda_p}. \tag{2.7}$$

The total stretch rate defined by eqn $(2.7)_1$ is the rate of change of length *measured per unit current length*, while the plastic stretch rate defined by $(2.7)_2$ is the rate of change of length due to only plastic deformation, *measured per unit length in the intermediate configuration*, $\mathscr{C}_p$. Since these quantities are not referred to the same configuration, their difference presents an awkward quantity and, in general, would involve both the rate of elastic stretch and that of plastic stretch. In fact, if eqn $(2.7)_2$ is taken to define the plastic stretch rate, then eqn $(2.7)_3$ provides no useful information.

The difficulty can be resolved very easily. We observe that since $(\mathrm{d}s_p)^{\cdot}$ is in fact the rate of change of length due to purely plastic deformation, then the rate of change of length due to only elastic deformation is given by $(\mathrm{d}s_e)^{\cdot}=(\mathrm{d}s)^{\cdot}-(\mathrm{d}s_p)^{\cdot}$, *and since the rates are involved, this is an exact result.* Hence, one has the following *exact* and useful rate decompositions:

$$\frac{(\mathrm{d}s)^{\cdot}}{\mathrm{d}s}=\frac{(\mathrm{d}s_e)^{\cdot}}{\mathrm{d}s}+\frac{(\mathrm{d}s_p)^{\cdot}}{\mathrm{d}s},$$

$$\frac{(\mathrm{d}s)^{\cdot}}{\mathrm{d}s_p}=\frac{(\mathrm{d}s_e)^{\cdot}}{\mathrm{d}s_p}+\frac{(\mathrm{d}s_p)^{\cdot}}{\mathrm{d}s_p},$$

$$\frac{(\mathrm{d}s)^{\cdot}}{\mathrm{d}S}=\frac{(\mathrm{d}s_e)^{\cdot}}{\mathrm{d}S}+\frac{(\mathrm{d}s_p)^{\cdot}}{\mathrm{d}S}. \tag{2.8}$$

The first decomposition measures all the rates per unit current length, and therefore, for actual numerical calculations, represents the most useful (exact) results in finite elastoplasticity. The second decomposition, $(2.8)_2$, which is obtained from $(2.8)_1$ by multiplying the latter by Λ_e, measures the elastic stretch rate per unit length in the *unstressed* intermediate configuration $\mathscr{C}_p$. Therefore, it represents a logical quantity for expressing the corresponding stress rate; however, in application, the calculation of $\mathrm{d}s_p$ is a complicated task. Finally, the last decomposition, $(2.8)_3$, which is obtained from $(2.8)_1$ by multiplying the latter by Λ, measures stretch rates per unit length in the undeformed original configuration $\mathscr{C}_0$; it is a Lagrangian (exact) rate decomposition. We note that the decomposition $(2.8)_1$ is the basis of the rate constitutive formulation for finite elastoplasticity presented by Hill[7].

There are additional complications pertaining to Λ_e as a measure of "elastic" stretch. In fact, a strict and logical use of Λ_e as the elastic stretch leads to contradiction, as it does not remain constant if the considered material neighborhood is subjected to additional ***infinitesimal*** purely plastic deformations.

To see this, consider an additional infinitesimal *purely plastic* deformation which changes the length of this element in configuration $\mathscr{C}_p$ from $\mathrm{d}s_p$ to $\mathrm{d}s_p + \epsilon$. Since this change is purely plastic, the corresponding "elastic recovery", i.e. the change in length resulting from the elastic unloading, must remain constant, which means that $\mathrm{d}s - \mathrm{d}s_p = \text{constant}$.† Hence $\mathrm{d}s$ must change to $\mathrm{d}s + \epsilon$, and the elastic stretch $(2.4)_3$ becomes $(\mathrm{d}s + \epsilon)/(\mathrm{d}s_p + \epsilon)$. Since $|\epsilon/\mathrm{d}s_p| \ll 1$ and $|\epsilon/\mathrm{d}s| \ll 1$, we obtain

$$\frac{\mathrm{d}s + \epsilon}{\mathrm{d}s_p + \epsilon} \simeq \Lambda_e + \frac{\epsilon}{\mathrm{d}s_p}(1 - \Lambda_e), \tag{2.9}$$

where $\Lambda_e = \mathrm{d}s/\mathrm{d}s_p$. This shows that *the elastic stretch* Λ_e defined by $(2.4)_3$ is not independent of additional purely plastic infinitesimal deformations. In fact, if the material element has been extended so that $\Lambda_e > 1$, and if the additional infinitesimal plastic deformation is also in extension, so that $\epsilon > 0$, eqn (2.9) leads to contradiction, since it indicates that the new value of "elastic stretch" (defined by eqn 2.4_3) is smaller than its value prior to the additional infinitesimal purely plastic extension. No such contradiction would result if the stretch rates are referred to and measured per unit length in the *same* configuration, as in eqns (2.8).

We note that the "plastic stretch," Λ_p, defined by $(2.4)_2$, remains constant when any additional purely elastic deformation is considered. Because, for such a purely elastic deformation, $\mathrm{d}s_p$ remains constant, and hence $\Lambda_p = \mathrm{d}s_p/\mathrm{d}S$ remains constant.

It is therefore reasonable to seek to obtain an "elastic" stretch measure, different from Λ_e, which is counterpart of Λ_p in the sense that it remains constant for all ***additional purely plastic deformations***.

This is easily done if one integrates the exact expression $(2.8)_3$ with respect to time, and notes the initial condition‡ $\mathrm{d}s_p = \mathrm{d}s_e = \mathrm{d}S$ in $\mathscr{C}_0$. In this manner, one obtains,

$$\frac{\mathrm{d}s}{\mathrm{d}S} = \frac{\mathrm{d}s_e}{\mathrm{d}S} + \frac{\mathrm{d}s_p}{\mathrm{d}S} - 1. \tag{2.10}$$

If we now set

$$\bar{\Lambda}_e = \frac{\mathrm{d}s_e}{\mathrm{d}S}, \tag{2.11}$$

we obtain from (2.10) and $(2.4)_{1,2}$

$$\Lambda = \bar{\Lambda}_e + \Lambda_p - 1. \tag{2.12}$$

To understand the physical meaning of the new elastic stretch, $\bar{\Lambda}_e$, we note that, upon unloading, the elastic recovery is given by $\mathrm{d}u_e = \mathrm{d}s - \mathrm{d}s_p$. Hence the length of the element $\mathbf{dX}$ would be given by

$$\mathrm{d}s_e = \mathrm{d}S + \mathrm{d}u_e = \mathrm{d}S + (\mathrm{d}s - \mathrm{d}s_p) \tag{2.13}$$

if this element were to deform purely elastically from its initial length $\mathrm{d}S$ by an amount equal to

†Note that, to the first order of approximation in ϵ, the elastic recovery must not be coupled with ϵ, (i.e. $(\mathrm{d}s_e)^{\cdot}$ and $(\mathrm{d}s_p)^{\cdot}$ must be uncoupled), otherwise the entire notion of "elastic unloading and decomposition to elastic and plastic parts of deformation" becomes meaningless, as perhaps may be the case for most *real* materials.

‡The quantity $\mathrm{d}s_e$ is defined by $\mathrm{d}s_e = (\mathrm{d}s - \mathrm{d}s_p) + \mathrm{d}S$ (see eqn 2.13).

the elastic recovery du_e. Thus, the elastic stretch relative to the initial configuration $\mathscr{C}_0$ becomes

$$\bar{\Lambda}_e = \frac{dS + (ds - ds_p)}{dS} = \frac{ds_e}{dS} = \Lambda - \Lambda_p + 1 \tag{2.14}$$

which is (2.12). Clearly enough, this measure remains constant for all additional purely plastic deformations, because $ds - ds_p = du_e$ remains constant for such deformations.

In conjunction with (2.11), we introduce a new "plastic stretch" defined by

$$\bar{\Lambda}_p = \frac{ds}{ds_e}, \tag{2.15}$$

and obtain the following decomposition of the total stretch:

$$\Lambda = \bar{\Lambda}_p \bar{\Lambda}_e. \tag{2.16}$$

This is the counterpart of the decomposition (2.5). Moreover, from (2.16) and (2.12) one obtains,

$$\Lambda^{-1} = \bar{\Lambda}_p^{-1} + \Lambda_e^{-1} - 1 \tag{2.17}$$

which is the Eulerian counterpart of (2.12).

In closing this section we emphasize that both the total decompositions (2.5) and (2.16) are formal, and their practical usefulness remains to be established. The decomposition (2.5) due to Lee[1] measures the total elastic stretch per unit length in the intermediate unstressed (but plastically deformed) configuration $\mathscr{C}_p$, and while it is a logical quantity for expressing the stress in constitutive relations, it leads to an awkward expression for the rate of elastic stretch. The quantity Λ_p, however, is a proper measure of the total plastic stretch (for a limited class of materials) and leads to a proper expression for the plastic stretch rate. In the decomposition (2.16), $\bar{\Lambda}_e$ leads to a proper expression for the elastic stretch rate. On the other hand, the quantity, $\bar{\Lambda}_p$, is the plastic stretch measured per unit length in a second intermediate configuration which is obtained by imposing only the elastic deformation on the undeformed material. Hence, a plastic stretch rate obtained on the basis of $\bar{\Lambda}_p$ will be an awkward quantity, and will not be independent of any additional infinitesimal purely elastic deformations.

We shall now proceed to the more general case.

3. HOMOGENEOUS DEFORMATIONS

Assume that $\mathscr{B}$ is homogeneous in $\mathscr{C}_0$, and that it undergoes a homogeneous deformation

$$\mathbf{x} = F\mathbf{X}, \quad F = F(t). \tag{3.1}$$

Upon unloading, configuration $\mathscr{C}_p$ is attained (see Fig. 1), where

$$\mathbf{p} = F^p\mathbf{X}, \quad F^p = F^p(t). \tag{3.2}$$

The mapping from $\mathscr{C}_p$ to $\mathscr{C}$ is elastic and homogeneous, so that

$$\mathbf{x} = F^e\mathbf{p}, \quad F^e = F^e(t). \tag{3.3}$$

From (3.1) to (3.3), one obtains

$$F = F^eF^p \tag{3.4}$$

which has been discussed by Lee[1] (see also eqn 2.5).

With no loss in generality, one may assume that the determinants of F^e and F^p are both positive and finite. Nevertheless the decomposition (3.4) cannot, in general, be unique. For

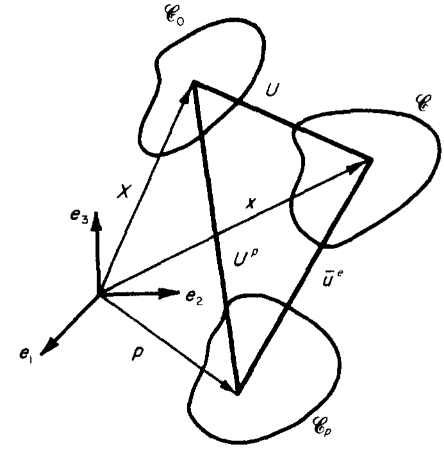

Fig. 1.

example, if we apply a rigid rotation to $\mathscr{B}$ in $\mathscr{C}_p$, we obtain $\overset{*}{\mathscr{C}}_p$ with the corresponding mapping

$$\mathbf{p}^* = R\mathbf{p}, \quad \det R = +1, \quad R^{-1} = R^T. \tag{3.5}$$

Equations (3.2)–(3.4) then become

$$\begin{aligned} &\mathbf{p}^* = (RF^p)\mathbf{X} = \overset{*}{F}{}^p\mathbf{X}, \quad \mathbf{x} = (F^eR^T)\overset{*}{\mathbf{p}} = \overset{*}{F}{}^e\mathbf{p}^*, \\ &\overset{*}{F}{}^p = RF^p, \quad \overset{*}{F}{}^e = F^eR^T, \quad F = \overset{*}{F}{}^e\overset{*}{F}{}^p. \end{aligned} \tag{3.6}$$

Instead of the pure rotation R, one may consider a general homogeneous deformation H with $0 < \det H < \infty$. But then the corresponding intermediate configuration will either be stressed or the "unloading" from $\mathscr{C}$ will involve an *additional* plastic flow. ***Thus, on physical grounds, it appears reasonable to assume that all intermediate configurations of $\mathscr{B}$, which are obtained upon unloading and without any additional plastic flow, differ from each other by rigid body rotations only.***†

A material element $\mathbf{dX}$ with the squared length $\mathbf{dX} \cdot \mathbf{dX} = (\mathrm{d}S)^2$, is deformed plastically by mapping (3.2) into $\mathbf{dp} \cdot \mathbf{dp} = (\mathrm{d}s_p)^2$, and one has

$$(\mathrm{d}s_p)^2 = \mathbf{dX}^T F^{pT} F^p \,\mathbf{dX} = \mathbf{dX} \cdot C^p \,\mathbf{dX}, \tag{3.7}$$

where

$$C^p = F^{pT}F^p \tag{3.8}$$

is the corresponding Green's deformation tensor; i.e. the metric tensor on the convected coordinates X_A in configuration $\mathscr{C}_p$. Thus the Lagrangian strain

$$E^p = \frac{1}{2}[C^p - I] \tag{3.9}$$

does indeed measure the plastic strain (see Green and Naghdi[8]). In fact $(\mathrm{d}s_p)^2 - (\mathrm{d}S)^2 = \mathbf{dX} \cdot [C^p - I]\,\mathbf{dX}$, and the corresponding stretch-squared is given by $(\mathrm{d}s_p/\mathrm{d}S)^2 = \Lambda_p{}^2 = \mathbf{N} \cdot C^p\mathbf{N}$,$\mathbf{N}$ being the unit vector along $\mathbf{dX}$. Note that the imposed rotation R in (3.6) does not affect C^p, i.e. $\overset{*}{C}{}^p = C^p$.

If one chooses the intermediate configuration $\mathscr{C}_p$ as the reference one, then $C^e = F^{eT}F^e$ defines the metric tensor in $\mathscr{C}$ relative to $\mathscr{C}_p$, and one has $(\mathrm{d}s)^2 = \mathbf{dx} \cdot \mathbf{dx} = \mathbf{dp} \cdot C^e\,\mathbf{dp}$. On the other hand, with $\mathscr{C}_0$ as the reference configuration, C^e is not a good measure of elastic deformation, since it

†Hence, by the polar decomposition, rotation can be eliminated completely.

depends on the rotation R that may be imposed on $\mathscr{C}_p$, i.e. $\mathring{C}^e = RC^eR^T$, as has been noted by Green and Naghdi[8].

Let us now look at the quantity

$$C' = C - C^p, \tag{3.10}$$

and observe that

$$(\mathrm{d}s)^2 - (\mathrm{d}s_p)^2 = \mathrm{d}\mathbf{X} \cdot C' \,\mathrm{d}\mathbf{X} \tag{3.11}$$

which shows that C' measures the *change* in the squared length from $\mathscr{C}_p$ to $\mathscr{C}$. *While* C *and* C^p *are proper metric tensors*,† C′ *is not*; for example C' is not positive-definite. Moreover, from (3.11) it is seen that, unlike C and C^p, the normal component of C' in a given direction $\mathbf{N}$ does not give the corresponding, i.e. "elastic", stretch-squared of the element initially in that direction, but rather it gives the total stretch-squared minus the plastic stretch-squared, i.e.

$$\mathbf{N} \cdot C'\mathbf{N} = \Lambda^2 - \Lambda_p{}^2 = \left(\frac{\mathrm{d}s}{\mathrm{d}S}\right)^2 - \left(\frac{\mathrm{d}s_p}{\mathrm{d}S}\right)^2. \tag{3.12}$$

Hence C' does not possess properties similar to those of C^p, and therefore should not be given the same significance. The same remarks apply to the quantity $E' = E - E^p$, which should not be viewed as the "elastic" part of the Lagrangian strain.

In view of the above remarks, it is natural to seek to obtain an "elastic Green's deformation tensor" which is the counterpart of the plastic Green's deformation tensor C^p, and hence is a proper metric tensor when $\mathscr{C}_0$ is used as the reference configuration.

To this end, observe from eqns (3.1)–(3.3) that the displacement field $\mathbf{U}$ can be decomposed into $\mathbf{U}^p$ and $\bar{\mathbf{u}}^e$ (which, for the lack of a better terminology, will be called "plastic" and "elastic," respectively) as follows:

$$\mathbf{U} = \mathbf{U}^p + \bar{\mathbf{u}}^e, \tag{3.13}$$

where

$$\mathbf{U} = (F - I)\mathbf{X}, \quad \mathbf{U}^p = (F^p - I)\mathbf{X}, \quad \bar{\mathbf{u}}^e = (F^e - I)\mathbf{p}. \tag{3.14}$$

We now substitute into $(3.14)_3$ for $\mathbf{p}$ from (3.3), and in view of (3.4) obtain

$$\begin{aligned}\bar{\mathbf{u}}^e &= (F^e - I)F^p\mathbf{X} \\ &= (F - F^p)\mathbf{X} \equiv (\bar{F}^e - I)\mathbf{X},\end{aligned} \tag{3.15}$$

where $\bar{\mathbf{F}}^e$ is the "elastic deformation gradient" which maps *elastically* the initial configuration $\mathscr{C}_0$ into an intermediate (elastic) configuration $\mathscr{C}_e$ (see Fig. 2), i.e.

$$\boldsymbol{\eta} = \bar{F}^e\mathbf{X}, \quad \bar{F}^e = \bar{F}^e(t). \tag{3.16}$$

Hence we have the following exact relation between $\bar{\mathbf{F}}^e$ and $\mathbf{F}^p$ (compare with 2.12):

$$F = \bar{F}^e + F^p - I. \tag{3.17}$$

The mapping from $\mathscr{C}_e$ to $\mathscr{C}$ then is purely plastic,

$$\mathbf{x} = \bar{F}^p\boldsymbol{\eta}, \quad \bar{F}^p = \bar{F}^p(t), \tag{3.18}$$

†Represented as matrices here.

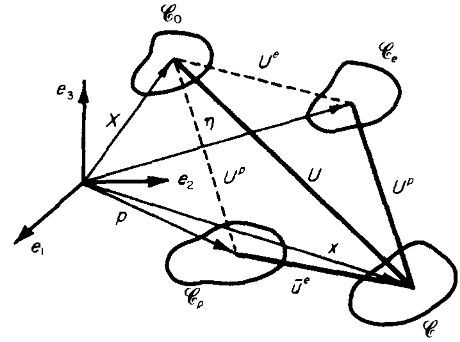

FIG. 2.

and one has (compare with 3.4 and 2.16)

$$F = \bar{F}^p \bar{F}^e. \tag{3.19}$$

From (3.17) and (3.19) it now follows that (compare with 2.17)

$$F^{-1} = F^{e^{-1}} + \bar{F}^{p^{-1}} - I. \tag{3.20}$$

While (3.17) uses $\mathscr{C}_0$ as the reference state (Lagrangian), decomposition (3.20) employs the final configuration for the reference one (Eulerian).

The metric tensor defined by

$$\bar{C}^e = \bar{F}^{e^T} \bar{F}^e \tag{3.21}$$

now is the counterpart of C^p, in the sense that the element dX is mapped *elastically* to $\mathrm{d}\boldsymbol{\eta}$, and we have

$$(\mathrm{d}s_e)^2 = \mathrm{d}\boldsymbol{\eta} \cdot \mathrm{d}\boldsymbol{\eta} = \mathrm{d}\mathbf{X} \cdot \bar{C}^e\, \mathrm{d}\mathbf{X}. \tag{3.22}$$

In particular, the elastic stretch-squared is given by

$$\Lambda_e{}^2 = \left(\frac{\mathrm{d}s_e}{\mathrm{d}S}\right)^2 = \mathbf{N} \cdot \bar{C}^e \mathbf{N}. \tag{3.23}$$

We note that the rigid rotation (3.5) changes F^p into $\overset{*}{F^p}$ and from (3.15) it follows that

$$(\overset{*}{F^e} - I)\overset{*}{F^p} = F - \overset{*}{F^p} = \overset{*}{\bar{F}^e} - I \tag{3.24}$$

which defines the manner by which $\overset{*}{\bar{F}^e}$ relates to $\overset{*}{F^p}$. Moreover, if $\bar{F}^e$ is replaced by $\overset{*}{\bar{F}^e} = \hat{R}\bar{F}^e$ where $\hat{R}$ represents a proper rotation matrix, we obtain $\overset{*}{\bar{C}^e} = \bar{C}^e$, so that $\bar{C}^e$ is a proper metric tensor for the "elastic" configuration $\mathscr{C}_e$ when $\mathscr{C}_0$ is used as the reference configuration.

It should be carefully noted that $\bar{F}^e$ is defined in terms of F^p and *after* the intermediate state $\mathscr{C}_p$ is obtained. That is, we define the "elastic" displacement field (for homogeneous deformation),

$$\mathbf{U}^e = (\bar{F}^e - I)\mathbf{X}, \tag{3.25}$$

such that

$$\mathbf{U} = \mathbf{U}^p + \mathbf{U}^e; \tag{3.26}$$

(see Fig. 2). Hence $\mathbf{u}^{\cdot} = \mathbf{U}^e$ and we have the decomposition (3.17). One may, of course, view

(3.17) as the *definition* of $\bar{F}^e$. At any rate, configuration $\mathscr{C}_e$ and hence $\bar{F}^e$ is obtained *after* the intermediate state $\mathscr{C}_p$ is *fixed*. Note that, for this homogeneous case in which $\mathscr{C}_p$ and $\mathscr{C}_e$ *are both compatible configurations*, decomposition (3.17) is obtained from (3.13) or (3.26) by taking partial derivatives with respect to **X**, i.e.

$$\frac{\partial U_A}{\partial X_B} = \frac{\partial U_A{}^p}{\partial X_B} + \frac{\partial U_A{}^e}{\partial X_B}. \tag{3.27}$$

[In the general case, however, the matrices F^p and $\bar{F}^e$ will not consist of partial derivatives. We note that even in such a case (3.17) applies (but locally), although it will no longer be the same as (3.27), since $\mathscr{C}_p$ and $\mathscr{C}_e$ will no longer each be a compatible state.] For the one-dimensional homogeneous case, this situation is shown in Fig. 3. Here the element of initial length L is stretched to the final length l, so that $F = l/L$. Upon unloading, length $l_p = L + U^p$ is attained, so that $F^p = l_p/L = (L + U^p)/L$. The elastic displacement is $\bar{u}^e = l - l_p$, so that $\bar{F}^e = (L + \bar{u}^e)/L = l_e/L$. Moreover, $\bar{F}^p = F\bar{F}^{e^{-1}} = l/l_e$. Clearly, we have $\bar{F}^e = l_e/L = F - F^p + 1 = (l - U^p - L + L)/L$.

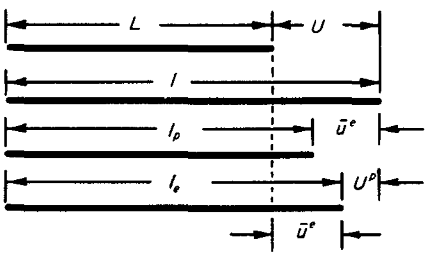

Fig. 3.

4. RATE OF CHANGE OF HOMOGENEOUS DEFORMATIONS

If one takes the material time derivative (i.e. time derivative with **X** held fixed) of both sides of (3.4), one obtains

$$\dot{F} = \dot{F}^e F^p + F^e \dot{F}^p. \tag{4.1}$$

Then, upon multiplication by F^{-1}, this yields

$$L \equiv \left[\frac{\partial v_a}{\partial x_b}\right] = \dot{F}F^{-1} = \dot{F}^e F^{e^{-1}} + F^e \dot{F}^p F^{p^{-1}} F^{e^{-1}}. \tag{4.2}$$

Lee[1] states that $L^e = \dot{F}^e F^{e^{-1}}$ is the elastic part and $L^p = \dot{F}^p F^{p^{-1}}$ is the plastic part of the velocity gradient. On the basis of this, he then concludes that "the velocity strains are not additive to give the total velocity strain, but for infinitesimal elastic strains $F^e \sim I$, the unit matrix, and additivity applies to some order of approximation". We shall show that this statement is not correct;† because the deformation rate tensor $D = (1/2)(L + L^T)$ can always be divided additively and exactly as‡

$$D = \hat{D}^p + \hat{D}^e \tag{4.3}$$

provided that the corresponding elastic and plastic deformation measures are referred to the same reference configuration. The situation here is exactly the same as the one-dimensional consideration presented at the end of Section 2. The elastic deformation gradient F^e, like the elastic stretch Λ_e in eqn $(2.4)_3$, ***does not remain constant when additional infinitesimal purely plastic deformations are superimposed on the body. Hence*** $\mathrm{L}^e = \dot{\mathrm{F}}^e \mathrm{F}^{e^{-1}}$ ***is not independent of the rate of plastic deformation.*** If instead of F^e we use $\bar{F}^e$ defined by (3.17), we then obtain (4.3) which is

†This has been shown in the case of the one-dimensional deformation in Section 2 (see eqns 2.6–2.8 and the corresponding discussions).

‡These tensor quantities are represented here by matrices.

exact, and which constitutes the basis of Hill's[7] rate formulation of finite deformation elastoplastic constitutive relations. To see this, take the material time derivative of both sides of (3.17) to obtain

$$\dot{F} = \dot{F}^e + \dot{F}^p. \tag{4.4}$$

Hence

$$\dot{F}F^{-1} = \dot{F}^e F^{-1} + \dot{F}^p F^{-1} \equiv \hat{L}^e + \hat{L}^p. \tag{4.5}$$

To see the meaning of each term in (4.5), note that the deformation is homogeneous and hence one can write (4.5) as

$$\frac{\partial \dot{x}_a}{\partial x_b} = \frac{\partial \dot{\eta}_a}{\partial X_A}\frac{\partial X_A}{\partial x_b} + \frac{\partial \dot{p}_a}{\partial X_A}\frac{\partial X_A}{\partial x_b} = \frac{\partial \dot{\eta}_a}{\partial x_b} + \frac{\partial \dot{p}_a}{\partial x_b}. \tag{4.6}$$

Hence we have $\hat{L}^e = [\partial\dot{\eta}_a/\partial x_b]$ and $\hat{L}^p = [\partial\dot{p}_a/\partial x_b]$, and the symmetric part of 4.5 yields (4.3). From (4.6) one also has

$$\frac{\partial \dot{x}_a}{\partial p_b} = \frac{\partial \dot{\eta}_a}{\partial p_b} + \frac{\partial \dot{p}_a}{\partial p_b} \tag{4.7}$$

which is the rate decomposition referred to the intermediate configuration $\mathscr{C}_p$; compare with $(2.8)_2$.

We therefore see that $\hat{D}^p$ in (4.3) is not the symmetric part of L^p, nor is $\hat{D}^e$ the symmetric part of L^e; in fact, contrary to the statement by Lee[1], L^e is *not* a measure of the pure "elastic velocity gradient", because *it does not remain constant for a purely plastic rate of deformation.* These facts are further discussed below.

For the *homogeneous case*, we have $F^e = [\partial x_a/\partial p_b]$, $F^p = [\partial p_a/\partial X_A]$ and hence (4.1) becomes

$$\frac{\partial \dot{x}_a}{\partial X_A} = \left(\frac{\partial x_a}{\partial p_b}\right)^{\cdot} \frac{\partial p_b}{\partial X_A} + \frac{\partial x_a}{\partial p_b}\frac{\partial \dot{p}_b}{\partial X_A}, \tag{4.8}$$

where, since dot denotes the material time derivative, it commutes with $\partial/\partial X_A$ but *not* with $\partial/\partial p_b$. Hence (4.2) becomes

$$\frac{\partial \dot{x}_a}{\partial x_c} = \frac{\partial v_a}{\partial x_c} = \left(\frac{\partial x_a}{\partial p_b}\right)^{\cdot} \frac{\partial p_b}{\partial x_c} + \frac{\partial x_a}{\partial p_b}\frac{\partial \dot{p}_b}{\partial x_c}. \tag{4.9}$$

But $\mathbf{x} = \bar{\mathbf{u}}^e + \mathbf{p}$, and the last term equals $(\partial\dot{p}_a/\partial x_c) + (\partial\bar{u}_a{}^e/\partial p_b)(\partial\dot{p}_b/\partial x_c)$ so that

$$F^e L^p F^{e^{-1}} = \left[\frac{\partial \dot{p}_a}{\partial x_c}\right] + \left[\frac{\partial \bar{u}_a{}^e}{\partial p_b}\frac{\partial \dot{p}_b}{\partial x_c}\right] = \hat{L}^p + \left[\frac{\partial \bar{u}_a{}^e}{\partial p_b}\right]\hat{L}^p \tag{4.10}$$

which is independent of the *elastic rate* of deformation, and hence represents a proper plastic deformation rate. From (4.9) it follows that

$$L^e = \dot{F}^e F^{e^{-1}} = L - \hat{L}^p - \left[\frac{\partial \bar{u}_a{}^e}{\partial p_b}\right]\hat{L}^p = \hat{L}^e - \left[\frac{\partial \bar{u}_a{}^e}{\partial p_b}\right]\hat{L}^p, \tag{4.11}$$

which is *not* independent of the plastic rate of deformation. Note that upon addition, the second (nonlinear) terms in (4.10) and (4.11) cancel out, resulting in the decomposition (4.6). Note also that a similar difficulty exists for the general (macroscopically) nonhomogeneous case.

Consider now a convected coordinated system which in the original reference configuration $\mathscr{C}_0$ coincides with the fixed rectangular Cartesian system. The covariant base vectors of the convected system then are $\mathbf{g}_A = x_{a,A}\mathbf{e}_a$, and the contravariant base vectors are defined by $\mathbf{g}^A \cdot \mathbf{g}_B = \delta_B{}^A$, where $\delta_B{}^A$ is the Kronecker delta. From (2.1) and (2.2) we then obtain

$$\dot{E}_{AB} = D_{ab}x_{a,A}x_{b,B} \tag{4.12}$$

which shows that $\dot{E}_{AB}$ are the covariant components of the deformation rate tensor in the convected coordinate system. Hence, the decomposition (4.3) can be written as

$$\dot{E}_{AB} = \dot{E}^e_{AB} + \dot{E}^p_{AB}, \tag{4.13}$$

where

$$\dot{E}^e_{AB} = \hat{D}^e_{ab}x_{a,A}x_{b,B}, \quad \dot{E}^p_{AB} = \hat{D}^p_{ab}x_{a,A}x_{b,B}. \tag{4.14}$$

Therefore, while one *cannot* additively decompose the Lagrangian strain, the corresponding strain rate can be decomposed in that manner, as in (4.13).

If $\dot{w}$ is the rate of work per unit mass, then one obtains

$$\dot{w} = \frac{1}{\rho}T_{ab}D_{ab} = \frac{1}{\rho_0}\tau^{AB}\dot{E}_{AB}, \tag{4.15}$$

where $\mathcal{T} = T_{ab}\mathbf{e}_a\mathbf{e}_b$ is the Cauchy stress tensor, and $\boldsymbol{\tau} = (\rho_0/\rho)\mathcal{T} = \tau^{AB}\mathbf{g}_A\mathbf{g}_B$ is the Kirchhoff stress tensor; ρ_0 and ρ being the initial and the current mass-densities, respectively.

The elastic and plastic rates of work are then defined by

$$\dot{w}^e = \frac{1}{\rho}T_{ab}\hat{D}^e_{ab} = \frac{1}{\rho_0}\tau^{AB}\dot{E}^e_{AB},$$

$$\dot{w}^p = \frac{1}{\rho}T_{ab}\hat{D}^p_{ab} = \frac{1}{\rho_0}\tau^{AB}\dot{E}^p_{AB}, \tag{4.16}$$

and we must have

$$\dot{w}^p \geq 0, \tag{4.17}$$

where the equality occurs for elastic loading and unloading. There are a number of different ways by which $\dot{w}^e$ and $\dot{w}^p$ can be made explicit. This then relates to the manner by which the constitutive relations for elastoplastic bodies are formulated. We shall not discuss this point here, but simply point out that, for practical applications, Hill's rate formulation, (see for example, Hill[7]), perhaps presents considerable advantages (see Refs. [3–5] for further discussions). A careful discussion of various aspects of the thermodynamics of plasticity is given by Mandel[2] where reference to the related earlier works can also be found.

Having established the compatibility of the finite decomposition (3.4) and the rate decomposition (4.3), we note that either the constitutive theory of Hill or that of Mandel can then be used, depending on the class of materials and on the considered problem. However, it should be kept in mind that a physically meaningful constitutive relation must necessarily be formulated on the basis of a model which adequately accounts for the relevant dominant micromechanics of the actual material that is being studied.

5. NONHOMOGENEOUS DEFORMATIONS

For a general case of (macroscopically) nonhomogeneous deformation we consider a typical particle $\mathbf{X}_0$ and confine attention to such a small neighborhood, $N(\mathbf{X}_0)$, of this particle that the corresponding deformation can be regarded homogeneous there. We then set $F(t) = F(\mathbf{X}_0, t)$, and $\mathrm{d}\mathbf{X} = \mathbf{X} - \mathbf{X}_0$, where $\mathbf{X}$ is in $N(\mathbf{X}_0)$. The element $\mathrm{d}\mathbf{X}$ is now mapped into $\mathrm{d}\mathbf{x}$, and we have

$$\mathrm{d}\mathbf{x} = F\,\mathrm{d}\mathbf{X}, \quad \mathrm{d}\mathbf{p} = F^p\,\mathrm{d}\mathbf{X}, \quad \mathrm{d}\mathbf{x} = F^e\,\mathrm{d}\mathbf{p}, \tag{5.1}$$

so that $F = F^eF^p$, where $F^p = F^p(\mathbf{X}_0, t)$ and $F^e = F^e(\mathbf{X}_0, t)$. If $\mathbf{U}_0$ is the displacement of $\mathbf{X}_0$, upon unloading, this material point moves by $\bar{\mathbf{u}}_0{}^e$ and we have $\mathbf{U}_0 = \mathbf{U}_0{}^p + \bar{\mathbf{u}}_0{}^e$. A similar equation applies to the material point $\mathbf{X}$. Hence, upon subtraction, for the element $\mathrm{d}\mathbf{X}$, we can write

$$\mathrm{d}\mathbf{U} = \mathrm{d}\mathbf{U}^p + \mathrm{d}\bar{\mathbf{u}}^e, \tag{5.2}$$

where $\mathrm{d}\mathbf{U} = (F - I)\,\mathrm{d}\mathbf{X}$, $\mathrm{d}\mathbf{U}^p = (F^P - I)\,\mathrm{d}\mathbf{X}$, and $\mathrm{d}\bar{\mathbf{u}}^e = (F^e - I)\,\mathrm{d}\mathbf{p} = (F^e - I)\,F^p\,\mathrm{d}\mathbf{X} = (\bar{F}^e - I)\,\mathrm{d}\mathbf{X}$. From this last equation, we obtain the decomposition (3.17) which, however, has now a "local" significance only. Continuing to confine attention to the neighborhood $N(\mathbf{X}_0)$, we define the locally elastic mapping

$$\mathrm{d}\boldsymbol{\eta} = \bar{F}^e\,\mathrm{d}\mathbf{X}, \tag{5.3}$$

where

$$\bar{F}^e = F - F^p + I, \quad F = \bar{F}^p\bar{F}^e. \tag{5.4}$$

We therefore see that eqns (3.1)–(3.6) retain their validity provided that $\mathbf{X}$ is replaced by $\mathrm{d}\mathbf{X}$, $\mathbf{x}$ by $\mathrm{d}\mathbf{x}$, $\mathbf{p}$ by $\mathrm{d}\mathbf{p}$, and $\boldsymbol{\eta}$ by $\mathrm{d}\boldsymbol{\eta}$; eqn (3.27) no longer has any significance. In a similar way, (4.1)–(4.5) and (4.12)–(4.17) remain valid, together with the corresponding discussions and comments. We therefore see that the decompositions (3.4) and (3.19) remain valid for the nonhomogeneous case, but F^e, F^p, $\bar{F}^e$, and $\bar{F}^p$ must be interpreted in a local manner; decompositions (3.17) and (3.20) retain their validity. The rate decompositions (4.3), (4.5), and (4.13) also maintain their effectiveness, but interpretation (4.6), the statement that follows it, and other explanations in terms of partial differentiations must be discarded; however, the comments pertaining to L^e and L^p are still correct.

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